Free Products

Let G be a group and $G_1, G_2, \dots, G_n$ subgroups of G that generate G. i.e. for $x \in G$, we can write $x = x_1 x_2 \cdots x_m$ where each x_i is in some G_j. $(x_1, \dots, x_m)$ is called a word of length m in $G_1, \dots, G_n$ that represents x. ($( )$ is a word of length 0 representing 1)

If x_i and x_{i+1} are in the same G_j, then $(x_1, \dots, x_i x_{i+1}, \dots, x_m)$ is a word of length $m-1$ that represents x.

If x is represented by $(x_1, \dots, x_m)$ where no G_j contains two consecutive elements, then $(x_1, \dots, x_m)$ is a reduced word.

Ex: Let D_8 be the dihedral group of order 8, and $G_1 = \{1, r, r^2, r^3\}$, $G_2 = \{1, f\}$. Then any element $x \in D_8$ can be represented by (r^i, f^j) and by (f^k, r^l), some i, j, k, l.

Def: Let G be a group and $\{G_1, \dots, G_n\}$ a family of subgroups that generates G, s.t. $G_j \cap G_k = \{1\}$ if $j \neq k$. Then G is the free product of G, denoted $G = G_1 * \cdots * G_n$ if $\forall$ $x \in G$ there is only one reduced word that represents x.

(This definition can be generalized to an arbitrary collection of subgroups)

We can also define free groups functorially, by a universal property. This definition is not as useful when working w/ explicit examples, but it is very important when proving things (just like the univ. property of quotient spaces).

Functorial def: Let $\{G_\alpha\}$ be a family of subgroups of G. G is the <u>free product</u> of the G_α if for any group H and homomorphisms $h_\alpha : G_\alpha \to H$ there is a unique homomorphism $h : G \to H$ s.t. the diagram

$$\begin{tikzcd} G_\alpha \arrow[d, hook] \arrow[dr, "h_\alpha"] & \\ G \arrow[r, "h"'] & H \end{tikzcd}$$

commutes $\forall$ α.

Ex: We already know that D_8 is not the free product of $G_1 = \langle r \rangle$ and $G_2 = \langle f \rangle$. Let's check via universal property.

Let $H = G_1 \times G_2$ and $h_1 : G_1 \to H$, $h_2 : G_2 \to H$ the natural inclusions.

Then if $\exists$ a map $h : D_8 \to H$, we know $rf = fr^{-1}$; so

$$\begin{tikzcd} G_1 \arrow[d] \arrow[dr, "h_1"] & \\ D_8 \arrow[r, "h"] & H \\ G_2 \arrow[u] \arrow[ur, "h_2"'] & \end{tikzcd}$$

$h(r) = (r, 1)$, $h(f) = (1, f)$

$\Rightarrow$ $h(rf) = (r, f)$

and $h(fr^{-1}) = (r^{-1}, f) \neq (r, f)$,

so D_8 is not a free product of G_1 and G_2.

We can also take the free product of an arbitrary family of groups:

Def: Let $\{G_\alpha\}_{\alpha \in J}$ be a family of groups. If G is a group, and $i_\alpha : G_\alpha \to G$ an injection s.t. G is the free product of the $i_\alpha(G_\alpha)$, then G is called the <u>external free product</u> of the G_α.

Facts about external free products

1.) G always exists.

2.) G is unique up to isomorphism.

3.) Satisfies universal property given above.

4.) Elements in G are reduced words in the G_α.

Free Groups

Def: Let $\{a_\alpha\}$ be a set. Let $G_\alpha = \{ a_\alpha^n \mid n \in \mathbb{Z} \}$. G_α is a group w/ $a_\alpha^n \cdot a_\alpha^m = a_\alpha^{n+m}$.

The external free product of the groups $\{G_\alpha\}$ is the <u>free group on the elements</u> a_α.

Equivalently, if $\{a_\alpha\} \subseteq G$, and $\langle a_\alpha \rangle \cong \mathbb{Z}$, then if G is the free product of $\langle a_\alpha \rangle$, G is the free group on the elements a_α.

Exercise If F_1 is the free gp on generators $\{a_\alpha\}$ and F_2 the free gp on generators $\{b_\beta\}$, then $F_1 * F_2$ is the free gp on generators $\{a_\alpha\} \cup \{b_\beta\}$.

Presentations of groups

One way to describe an arbitrary group is to specify a set S of generators and a set R of relations.

Then G has presentation $\langle S \mid R \rangle$

If F is the free group on S, we get a surjection

$$h: F \to G,$$ and R (and their conjugates)

should generate N.

Ex: 1.) $\langle a \mid a^n = 1 \rangle \cong \mathbb{Z}/n\mathbb{Z}$

2.) $\langle r, f \mid r^n = 1, f^2 = 1, \underbrace{rf = fr^{-1}}_{\substack{\text{or rewrite as} \\ rfrf = 1}} \rangle \cong D_{2n}$

3.) $\langle a, b \mid ab = ba \rangle \cong \mathbb{Z} \times \mathbb{Z}$.